Lewin, Intervals, and Transformations: a Comment on Hook

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This paper takes up a small but significant point in Julian Hook's (2007) excellent "Cross-Type Transformations and the Path Consistency Condition." I admire Hook's work and have no fundamental disagreements with his goals, intuitions, or techniques. In particular, I applaud his attempts to move beyond some of the more restrictive aspects of David Lewin's formalism by abandoning the "path consistency condition" and permitting "cross-type transformations."

However, I do think that Hook's paper uses Lewinian terminology to advance a somewhat anti-Lewinian perspective, one that begins to replace group theory with less abstract mathematical tools. In particular, I believe that Hook's understanding of a "Generalized Interval System," while consistent with the letter of Lewin's definition, is at odds with their transformational spirit. I repeat that I do not mean this as a criticism: Hook may well be right to depart from Lewin. But the modesty of his rhetoric downplays one of the more radical implications of his own argument: that groups of transformations may not suffice for modeling musical intervals.

1. The canonical unlabeled interval group. A Lewinian "Generalized Interval System" (GIS) has three components: a set (or "space") of musical objects S, a group IVLS, and a bijective function int(x, y) mapping ordered pairs of elements in S into the group IVLS. In Chapter 3 of *Generalized Musical Intervals and Transformations,* Lewin notes that "in any GIS we can always use the intervals of the group IVLS to label the members of the space S by their respective intervals from an assumed referential object in S" (31)[1]. In other words, we can apply interval-names directly to objects themselves. For example, suppose our musical space consists of the twelve pitch classes {C, C♯, . . ., B} and our intervals consist of the group of integers modulo 12.[2] The function int(p, q) sends the ordered pair of pitch classes (p, q) into the interval x if pitch class q is x semitones above pitch class p. This allows us to use the integers modulo 12 to label pitch classes. Choosing C as our "reference pitch class," we can label every pitch class by its interval from C, giving the standard mapping: C = 0, C♯ = 1, D = 2, and so on. Crucially, however, this mapping requires the arbitrary choice of a "reference pitch class." Lewin takes great pains to emphasize that there is no privileged mapping between the integers {0, 1, . . ., 11} and the pitch classes {C, C♯, . . ., B}.[3]

Notice, however, that the argument also works the other way around. Because of the close relationship between objects and intervals, we can always use the names for the members of the set S to generate names for the intervals in a GIS. Suppose, for example, our musical objects are the twelve pitch classes. Suppose also that we have the elementary (cognitive, musical, analytical) ability to judge whether any two pairs of pitch classes are separated by the same or different intervals: that is, for any pitch classes p, q, r, and s, we can judge whether int(p, q) = int(r, s). Formally, we have a relation $\mathcal{R}$ such that (p, q)$\mathcal{R}$(r, s) if and only if int(p, q) = int(r, s). This information suffices to specify a group, whose

Thanks to Julian Hook and Steven Rings for helpful conversations.

1 Lewin develops this perspective at length in his 1977 article, "A Label-Free Development for 12-Pitch-Class Systems."

2 In this paper, I use curly braces {} for unordered collections and regular parentheses () for ordered collections.

3 In more modern mathematical terminology, the space of pitch classes is a torsor (or principal homogeneous space) of the group $\mathbb{Z}_{12}$.

elements are sets of ordered pairs of pitch classes related by a particular interval. For instance, we can represent the unison using the set

{(C, C), (C♯, C♯), (D, D), . . ., (B, B)}

and the interval between C and C♯ as:

{(C, C♯), (C♯, D), (D, E♭), . . ., (B, C)}

and so on. We can call these sets *unlabeled intervals:* the first is the unlabeled interval that connects every pitch class to itself, while the second is the unlabeled interval that links pitch class C to C♯, C♯ to D, etc.[4] (Unlabeled intervals are essentially permutation matrices, familiar from representation theory.) Furthermore, using the rule $\text{int}(p, q) + \text{int}(q, r) = \text{int}(p, s)$ we can determine how the intervals combine. The result is the *canonical unlabeled interval group* corresponding to the set S and the relation $\mathcal{R}$.[5] Note that here we have essentially reversed Lewin's procedure: rather than using intervals to name pitch classes, we are using sets of ordered pairs of pitch classes to name intervals. The resulting construction can be generalized to any GIS.

In avoiding interval labels, we have described intervals entirely by their transformational properties: rather than associating them with numbers, we simply describe *what they do* to the objects in the space. For this reason, the canonical unlabeled interval group can be understood to reveal the *transformational content* of a GIS. By contrast, the standard group of pc intervals reflects what Lewin calls a "Cartesian" perspective: here the elements of the group are represented by numbers that can be understood to describe the size and direction of the interval in question. It is an important theme of GMIT, expounded forcefully in the opening pages of Chapter 7, that these two perspectives are fundamentally equivalent.

But are they? We have two different GISes, one of which identifies pitch-class intervals with the group of integers modulo 12, and one of which identifies the set of pitch classes with transformations, represented by the canonical unlabeled interval group. The two groups in question are *isomorphic,* and in group theory they would typically be considered equivalent. (Group theorists often use phrases like "*the* group $\mathbb{Z}_{12}$" to refer to any group isomorphic to $\mathbb{Z}_{12}$.) But the two interval systems do not convey the same musical information. For just as there is no single natural mapping from the pitch classes to the integers mod 12, so too is there no single natural mapping from the *canonical unlabeled intervals* to the integers mod 12. This is because we are free to map the unlabeled interval

{(C, C♯), (C♯, D), (D, E♭), . . ., (B, C)}

to any of the four generators of $\mathbb{Z}_{12}$, 1, 5, 7 and 11. Since each choice gives us an equally good isomorphism from one group to the other, the two GISes are "equivalent" only up to this arbitrary choice. Musically, this means that it we cannot determine from the canonical unlabeled interval group whether we are measuring intervals in semitones or fifths, and whether we measure the interval (p, q) from p upward to the q or from p downward to q. Or to put the point more provocatively: if we consider intervals to be transformations, *it is no more natural to associate them with numbers than it is to associate pitch classes with numbers.*

We are thus at a crossroads. If we are "Cartesians," if we believe that intervals have particular sizes and directions, then we should label them numerically. From this perspective it makes a difference whether we use "1" or "5" to label

4 Note that I am deliberately avoiding describing this interval as "the ascending semitone," for reasons that will become apparent shortly.

5 The generic mathematical adjective "canonical" is not meant to evoke Lewin's use of the same word in Chapter 5 of GMIT; instead, it is meant to suggest that, for any GIS, there is a standard way to construct its "canonical unlabeled interval group."

the interval from C to C♯, as we thereby attribute different sizes to it. These numerical labels endow the intervals with additional metrical structure, over and above their purely group theoretical properties. If, on the other hand, we consider intervals truly to be transformations, then there is no need to label them with numbers at all. Instead, we can simply describe their effects on the objects in the space, using the canonical unlabeled interval group. From this perspective, there simply is no question whether semitones are smaller than fifths, or whether we measure intervals in an ascending or descending direction.[6] Thus the difference between the perspectives comes down to this: do the elements in the group IVLS have properties (such as magnitudes and directions) *that are not captured by* their transformational effects on the underlying musical objects? In other words, do we need to go beyond group theory to model musical intervals? The Cartesian says "yes" while the strict transformational theorist says "no."

2. Hook and cross-type transformations. Hook notes that there is often a theoretical and analytical need to compare objects and intervals of very different musical types. For instance, he observes that composers often map diatonic pitch sequences such as (C4, D4, E4) into chromatic sequences such as (C4, C♯4, D4), and vice versa. (A notable example occurs in the last movement of Berlioz's *Symphonie Fantastique,* when the diatonic fugue theme shifts into chromatic space.) But as Hook notes, it is not obvious, within Lewin's system, how to compare intervals in distinct GISes.

> Now consider a cross-type situation in which u and x inhabit one GIS (S_1) and v and y inhabit another (S_2). Here things are more complicated. The [. . .] two intervals int(u, x) and int(v, y) [. . .] cannot in general be compared because they belong to two different interval groups (13).

6 We can always assign "sizes" to various transformations. But then we have again left the realm of group theory, and endowed our musical intervals with additional structure. Furthermore, "transformations" are no longer simply functions, since they have additional structure that is not captured by their effect on objects in the space.

To see the force of this worry, let's consider the canonical unlabeled intervals corresponding to two particular GISes. In Hook's chromatic pitch space, the ascending semitone is represented by an infinite set of the form

{. . ., (B3, C4), (C4, C♯4), (C♯4, D4), . . .}

while in diatonic pitch space the ascending step would be represented by an infinite set of the form

{. . ., (B3, C4), (C4, D4), (D4, E4), . . .}.

These two infinite sets are not identical, and there is no single canonical isomorphism between the groups to which they belong. Instead there are two distinct isomorphisms: one that maps the ascending chromatic step into the ascending diatonic step, and one that maps it into the descending diatonic step. From the standpoint of group theory, there is no important difference between these. This is because the terms "ascending" and "descending" belong to more concrete branches of mathematics such as geometry.

Hook's resolution of this dilemma is both bold and un-Lewinian. He writes, "Often, however, two different GISes share the same interval group. [. . .] In such cases [. . .] the notion of 'interval preserving' is meaningful" (13). But here I think he makes a mistake. The mere fact that two GISes share the same interval group does *not* mean that their intervals can be meaningfully compared. This is because a single number can mean very different things in different contexts. Just because we measure weight and distance using numbers, this does not mean that one pound is meaningfully compared to one mile: the *numbers* may be identical, but they are embedded in unrelated measurement systems. In general, distinct GISes can measure properties as different as different as pounds as miles, and we have no guarantee that it is meaningful to compare their intervals, no matter how they are labeled.

There are two subtly different issues at play here. The first concerns Hook's phrase "the same group." Group theorists often use this phrase to mean "the same group *up to isomorphism,*" since they are typically concerned with structure ("group structure") that is shared by all isomorphic groups. But to the group theorist there is no unique or privileged way to compare the elements of isomorphic groups. (As we have seen, there are two isomorphisms between the group of chromatic intervals and the group of diatonic pitch intervals, and group theory is neutral between them.) Thus, when Hook talks about "the same group" he uses the phrase in a more particular sense—he means to be referring to the case where the two isomorphic groups label their elements using *precisely the same collection of symbols*. In so doing, I take him to be departing from the transformational approach, according to which it is *group structure* that is central to intervals' identity. Instead, he relies a more Cartesian perspective, according to which intervals have additional mathematical structure not determined by the transformations they induce on the space of musical objects—structure represented by the intervals' non-arbitrary numerical labels. We will return to this thought momentarily.

The second issue is more technical: Hook wants to use the terms "transposition-like" and "inversion-like" to describe mappings from one GIS to another ("GIS homomorphisms"); however, it is much more in the spirit of Lewin's enterprise to *relativize* these descriptions to GIS homomorphisms themselves. That is, given a GIS homomorphism, f: $S_1 \rightarrow S_2$, between musical spaces S_1 and S_2, we can define notions of "inter-GIS transposition and inversion" relative to that homomorphism. Suppose $s_1 \in S_1$ and $s_2 \in S_2$ are elements in two distinct musical spaces, and f: $S_1 \rightarrow S_2$, is a GIS homomorphism between the spaces to which they belong. Then s_1 relates to s_2 by "inter-GIS transposition i" if $f(T_i(s_1))=s_2$, and "inter-GIS inversion i" if $f(I_i(s_1))=s_2$. Using these definitions, we can determine whether arbitrary sequences of objects in the two GISes are related by (inter-GIS) transposition or inversion. As far as I can tell, this formalism is entirely adequate for the analytical applications in Hook's paper. Furthermore, it has the advantage of significantly greater generality: these notions of "inter-GIS transposition and inversion" apply to *any* GIS homomorphism, and not simply those between groups sharing the same intervals. Finally, the proposed formalism is explicit about the fact that GIS homomorphisms are what make cross-GIS comparison of intervals meaningful. Hook's alternative exploits what is potentially a merely *orthographical* fact: that two GISes happen to use the same symbols to label their intervals.

Let me now return to the Cartesian aspects of Hook's thinking. Hook evidently feels that the musical transformation that maps ascending diatonic steps into ascending chromatic steps is more "transposition-like" than the ("inversion-like") transformation that maps ascending diatonic steps into descending chromatic steps. I am sympathetic with this intuition: when we map ascending diatonic intervals to ascending chromatic intervals we are *preserving the direction of the intervals,* and this is one of the defining characteristics of transpositions generally. What I want to argue, however, is that we cannot capture this intuition when we restrict our attention the *group structure* of the relevant interval groups. And this means that formalizing Hook's intuition requires departing somewhat from Lewin's group-theoretical framework. If there is a guiding idea of GMIT it is that musical intervals can be understood as transformations. This principle is embodied in the mathematical requirement that intervals be understood as elements of groups—and specifically in the idea that it is the intervals' group structure, given by their transformational effects on the musical objects, that determines their identity. From this point of view, *there is no natural or privileged way to transport intervals from one musical space to another,* simply because the transformations on one set of musical objects do not apply to objects in other spaces.

To be fair, I think Lewin himself was sometimes less clear about these matters than he might have been. In "A Label-Free Development for 12-Pitch-Class Systems," and in Chapter 3 of GMIT, he writes as if intervals acquired their numerical names unproblematically—as if the standard "pitch class intervals" were straightforwardly associated with the integers from 0 to 11. (Curiously, however, he is very concerned to emphasize that *pitch classes* acquire their numerical names in an arbitrary fashion.) Chapter 2 of GMIT also assigns numerical labels to intervals, happily inhabiting the Cartesian framework. In these parts of GMIT, Lewin moves beyond group theory, endowing his intervals with metric structure over and above their purely transformational properties. The trouble starts only in Chapter 7 of GMIT, and specifically pp. 157–160, when Lewin asserts that Cartesian intervals-as-extensions are fundamentally equivalent to group-theoretical intervals-as-transformations. At a stroke, he seems to disavow the non-group-theoretical features of the Cartesian approach.[7]

Hook's paper forces us to confront these issues, since he explores transformations from one GIS to another. As I said, I applaud this aspect of his work, just as I applaud his rehabilitation of "path inconsistent" networks. I would just like to note that as we generalize Lewin's ideas, we may find that there are real differences between the Cartesian and "transformational" approaches. To the extent that we wish to assert that intervals have *size* and *direction*, we will want to model them with something more robust than group theory.

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7 Again, it could be that Lewin meant to endow transformations with additional metric structure, but then transformations are not simply functions.