

Mazzola's Counterpoint Theory

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Abstract: This paper critiques Guerino Mazzola's derivation of traditional counterpoint rules, arguing that those rules are not well-modeled by pitch-class intervals; that Mazzola's differential treatment of fifths and octaves is not supported musically or by traditional counterpoint texts; that Mazzola's specific calculations are not reproducible; that there are a number of intuitive considerations weighing against Mazzola's explanation; that the fit between theory and evidence is not good; and that Mazzola's statistical arguments are flawed. This leads to some general methodological reflections on different approaches to mathematical music theory, as well as to an alternative model of first-species counterpoint featuring the orbifold \mathbb{T}^2/S_2 .

Keywords: counterpoint, Guerino Mazzola, Fux, Palestrina, parallel fifths, antiparallel fifths, orbifolds, methodology of mathematical music theory.

1 Introduction

For an American music theorist, Guerino Mazzola's *The Topos of Music* (henceforth *Topos*) can be a forbidding book: dense, intricately systematic, and more complex in its mathematics than the writings of Allen Forte, John Clough, or David Lewin. And where American theorists can be somewhat apologetic in their invocations of advanced mathematics, offering simplified tutorials for untrained readers, Mazzola can seem almost aristocratic in his disdain for nonmathematicians. If you can't learn algebraic geometry, he sometimes seems to be saying, then you have no business trying to understand Mozart.

Confronted with this attitude, some theorists might feel tempted to dismiss *Topos* as a mathematical fantasy with no real-world relevance. But to do so is to ignore the fact that Mazzola makes claims of the utmost centrality to music theory. Perhaps foremost among these is the "Counterpoint Theorem," located 650 dense and technical pages into the tome: here Mazzola proposes that Fux's first-species counterpoint rules—rules which are basic to Renaissance composition and tonal pedagogy more generally—are in fact reflections of deep mathematical symmetries inherent in the twelve-tone universe. This proposal is important enough that the conscientious music theorist must take it seriously, even if that means working through some technical details.

The purpose of this paper is to use Mazzola's proposal to open a dialogue between American and European approaches to mathematical music theory. In doing so, I

want to make it clear that my intent is not hostile: every writer deserves critics, and every book—even a great book—deserves careful scrutiny.¹ Like many American readers, I am fascinated by *The Topos of Music*; indeed, my shock upon first encountering it is comparable to my shock on first encountering the philosophy of Hegel or the music of Cecil Taylor. Here, I felt, was something new, powerful, and yet utterly beyond my comprehension: was it a great intellectual achievement, or a majestic shaggy-dog story?

My initial motivation in writing this paper was to try to answer this question for myself. I am motivated to try to publish it by the thought that, although *Topos* has had many authors (twenty names are listed on the front cover), it has had relatively few critics—that is, independent, established, and probing music theorists who are able to follow its mathematical and musical details.² This is on one level unsurprising: if it takes an extraordinary degree of mathematical sophistication simply to read Mazzola’s book, it also takes a good deal of *musical* sophistication to evaluate its boldest theoretical claims. Thus an algebraic geometer, though perhaps comfortable with Mazzola’s mathematics, may find it difficult to evaluate the claims about Fux. (Of course, a specialist in Renaissance music has no hope of understanding Mazzola’s mathematical formalism.) For this reason, I suspect that there are relatively few readers who are truly able to evaluate this extraordinary and challenging work. All the more reason, then, for a theorist such as myself to give it a try.

2 Mazzola’s description of Fux’s first-species rules

Mazzola proposes to use contrapuntal symmetries to explain Fux’s first-species counterpoint rules. But before we can evaluate this proposal, we need to know whether he has characterized Fux’s rules correctly. (Roughly speaking, the rules are the data that his theory is meant to describe.) There are essentially three separate issues here. First, can Fux’s rules be usefully modeled by an approach that abstracts away from specific pitches in favor of pitch-class intervals? Second, if we accept this abstraction, is Mazzola justified in focusing on parallel fifths to the exclusion of octaves? And third, are Mazzola’s claims technically correct on their own terms?

Contemporary theorists may be somewhat surprised to learn that Fux’s discussion of first-species counterpoint contains no specific prohibition against parallel perfect intervals; instead, it forbids *all similar motion into perfect consonances*.³ Thus for Fux the successions (C4, G4)→(D4, A4) and (C4, E4)→(D4, A4) are equally bad, even though only the former involves forbidden parallels.⁴ Conversely, successions like (C4, G4)→(G3, D5) would seem to be acceptable, even though they involve “antiparallel fifths”—a fifth moving to a twelfth by contrary motion (or vice versa).

¹ It may be useful to know that Guerino encouraged me to write this paper, precisely in order to facilitate dialogue between different strands of mathematical music theory.

² For a valuable critique of Mazzola’s early work, see Roeder 1993.

³ Fux 1971, 22.

⁴ Throughout this paper, I use the notation $(x, y) \rightarrow (w, z)$ to indicate that note x moves to w and y moves to z . See Tymoczko 2011.

Judging by his explicit remarks, Fux also seems to permit “registral parallels” such as (C4, G4)→(C5, F5), in which crossed voices articulate a pair of perfect fifths in register.⁵ Thus he is concerned with a *very specific kind of melodic motion*, rather than a general harmonic situation in which one seven-semitone pitch-class interval follows another. Figure 1 shows that Fux is here being reasonably faithful to his Renaissance sources, which often contain registral parallels or “antiparallels” unusual in later music.⁶

It is therefore somewhat anachronistic to model counterpoint intervals modulo the octave. But since abstraction is the theorist’s prerogative, let’s permit Mazzola this relatively innocent simplification. The next question is why Mazzola permits progressions from unisons to octaves (or vice versa), while prohibiting the analogous progressions between fifths and twelfths. Most pedagogues instead treat perfect intervals as a unified class subject to the same fundamental constraints: in general, theorists either warn against *both* antiparallel octaves and fifths or permit both sorts of notion.⁷ It is unclear why Mazzola decides to prohibit only antiparallel fifths: in a move that is emblematic of *Topos*’s challenges, detailed explanations are found not in the book itself, but in an unpublished paper.⁸

One issue here is that it takes a significant amount of interpretation to extract a rigorous set of “counterpoint rules” from traditional pedagogical texts. For example, though neither Fux nor Jeppesen directly prohibits antiparallels, both prohibit unisons within the phrase while limiting the inter-voice distance to the interval of a tenth.⁹ As a result, antiparallels can occur only in unusual circumstances—octaves at the very beginning of a phrase, and fifths when there is unusually wide separation between the voices. Did Fux and Jeppesen mean to permit antiparallels in these specific circumstances, or were they more concerned with providing general rules that would help students write well? Did they reject antiparallel fifths *only* because of the wide registral separation, or were they simply uninterested in providing a comprehensive discussion of the multiple ways in which this particular progression was bad? The answers to these questions are by no means clear: one could certainly argue that Fux did reject first-species antiparallels, if only because there are no Fuxian examples in which they appear. (This is further supported by the fact that some Renaissance theorists allow antiparallels *only when there are a large number of voices*.¹⁰)

⁵ Fux explicitly permits voice crossings on page 36.

⁶ It is quite difficult to find passages such as Fig. 1a–b in Bach. I surmise that the increased sensitivity to these “quasi-parallels” testifies to an increasing awareness of the *harmonic* dimension of music, according to which chords are entities in themselves, ordered registrally, and not simply the byproducts of melodic lines.

⁷ See, e.g. Kostka and Payne, 2000, 84, or Gauldin 1985, 136.

⁸ Item 342 in Mazzola’s bibliography is “Mazzola G and Muzzolini D: Deduktion des Quintparallelenverbots aus der Konsonanz-Dissonanz-Dichotomie. Accepted for publication in: *Musiktheorie*, Laaber 1990.”

⁹ See Jeppesen 1939, 112 and Fux 1971, 38. Fux does not explicitly require that voices stay within a tenth, though all his examples obey this restriction.

¹⁰ Vincentino (1555, book 2, ch. 3, f. 41v.) allows anti-parallel fifths in five parts and anti-parallel octaves in eight parts. Note that this suggests that antiparallel octaves are *more* problematic than antiparallel fifths, contrary to Mazzola’s claim. Thanks here to Peter Schubert, who also supplied some of the examples in Figure 1.

However, this is not the only possible conclusion one could draw. The real point is that there is a huge gap between the pedagogical remarks in traditional counterpoint texts and rigorous models of the sort found in contemporary theory. All the more reason, then, to expect careful argumentation in favor of Mazzola's unusual decision to treat fifths and octaves so differently.

(a) (b) (c) (d) (e)

G C A E A E
C F D A D A

Fig. 1. Renaissance composers often articulated consecutive fifths registrally (*a–c*) or by contrary motion (*d–e*). (*a*) Palestrina, Missa “In te Domine speravi,” Credo; (*b*) Lassus, *Prophetiae Sybillarum*, Prologue, mm. 3–4; (*c*) Josquin “Gaude Virgo, Mater Christi,” m. 51, (*d*) Palestrina, Missa “Sanctorum meritis,” Agnus, m. 26, (*e*) Palestrina, Missa “Vestiva i Colli,” Kyrie, m. 78. Mazzola’s first-species counterpoint rules forbid antiparallel fifths such as those in the last two examples.

Finally, the technical details. Mazzola writes: “within an ecclesiastical mode, there are 287 a priori possible progressions. According to the consonance-dissonance counterpoint theorem, 37 of them are forbidden. Among them, 21 coincide with the 54 Fux-inadmissible progressions.” Despite having corresponded with Mazzola, I am able to reproduce only two of these four numbers. The number 287 refers to the number of distinct progressions of the form $(0, 7) \rightarrow (2, 5)$ where (*a*) the first dyad starts with 0; (*b*) each dyad is a consonance (that is: unison, third, perfect fifth, or sixth); and (*c*) the entire progression fits within some diatonic scale. Mazzola states that 54 of these 287 progressions are “Fux inadmissible.” I take this to mean that 54 progressions either involve parallel perfect fifths or contain a tritone in one of the voices.¹¹ However, by my count, the number of Fux-inadmissible progressions should be 55 or 65: 45 progressions contain tritones, 10 contain parallel fifths, and additional 11 contain parallel octaves, one of which also has tritone leaps (Figure 2).¹² (As noted above, I would argue in favor of the prohibition on antiparallel octaves, in which case there are 65 “Fux inadmissible” progressions.) Finally, Mazzola states that 21 of the 37 progressions forbidden by the Counterpoint Theorem are also Fux-inadmissible. The number 37 refers the number of diatonic successions in the bottom cell of the table on page 654.¹³ But by my count, the number of Fux-inadmissible progressions should be 19, with the remaining 17 not specifically prohibited in Renaissance music

¹¹ This is because “Fact 16” on page 657 of *Topos* states “in the reduced strict style, only the rule of forbidden fifth parallels and the tritone rules have an unrestricted validity.”

¹² Note that I do not consider repetitions, such as $(C4, G4) \rightarrow (C4, G4)$, to be parallel fifths. Mazzola forbids all such repetitions, no matter what interval they involve (Figure 3).

¹³ Mazzola’s table contains a few other progressions that articulate augmented triads, diminished seventh chords, and other nondiatonic sets.

(Figure 3).¹⁴ It is noteworthy that the two irreproducible numbers involve Fux's prohibitions, underscoring the point that it is difficult to agree about what they actually are.

(a)

The 21 diatonic progression-classes involving parallel fifths or octaves	
Octaves	Fifths
(0, 0)→(1, 1)	(0, 7)→(1, 8)
(0, 0)→(2, 2)	(0, 7)→(2, 9)
(0, 0)→(3, 3)	(0, 7)→(3, 10)
(0, 0)→(4, 4)	(0, 7)→(4, 11)
(0, 0)→(5, 5)	(0, 7)→(5, 0)
(0, 0)→(6, 6)*	(0, 7)→(7, 2)
(0, 0)→(7, 7)	(0, 7)→(8, 3)
(0, 0)→(8, 8)	(0, 7)→(9, 4)
(0, 0)→(9, 9)	(0, 7)→(10, 5)
(0, 0)→(10, 10)	(0, 7)→(11, 6)
(0, 0)→(11, 11)	

(b)

The 45 consonant diatonic progression-classes involving a tritone in at least one voice					
(0, 0)→(10, 6)	(0, 3)→(0, 9)	(0, 4)→(10, 10)	(0, 7)→(1, 1)	(0, 8)→(10, 2)	(0, 9)→(0, 3)
(0, 0)→(11, 6)	(0, 3)→(2, 9)	(0, 4)→(2, 10)	(0, 7)→(10, 1)	(0, 8)→(2, 2)	(0, 9)→(3, 3)
(0, 0)→(2, 6)	(0, 3)→(5, 9)	(0, 4)→(6, 2)	(0, 7)→(5, 1)	(0, 8)→(5, 2)	(0, 9)→(6, 2)
(0, 0)→(3, 6)	(0, 3)→(6, 1)	(0, 4)→(6, 6)	(0, 7)→(6, 2)	(0, 8)→(6, 1)	(0, 9)→(6, 6)
(0, 0)→(6, 1)	(0, 3)→(6, 10)	(0, 4)→(6, 9)	(0, 7)→(6, 6)	(0, 8)→(6, 10)	(0, 9)→(6, 9)
(0, 0)→(6, 10)	(0, 3)→(6, 3)	(0, 4)→(7, 10)	(0, 7)→(6, 9)	(0, 8)→(6, 3)	(0, 9)→(7, 3)
(0, 0)→(6, 2)	(0, 3)→(6, 6)			(0, 8)→(6, 6)	
(0, 0)→(6, 3)	(0, 3)→(9, 9)			(0, 8)→(7, 2)	
(0, 0)→(6, 6)*					
(0, 0)→(6, 9)					
(0, 0)→(9, 6)					

Fig. 2. (a) Diatonic progression classes containing parallel fifths or octaves. (b) Diatonic progression classes containing a tritone in at least one voice. A label like (0, 0)→(1, 1) refers to a class of progressions related by transposition; thus it could stand for (E, E)→(F, F) as well as (C, C)→(D♭, D♭). Each element of each class can be embedded within any diatonic scale by a suitable choice of pitch-class 0. Note that the starred progression contains both melodic tritones and parallel octaves.

¹⁴ Mazzola asserts that Fux prohibits the “battuta,” with (C, E)→(D, D) appearing on his list of forbidden progression. But this is a subtle issue: Aloysius, Fux’s counterpoint master, notes that the progression is traditionally prohibited when it moves from tenth to octave, but not when it moves from sixth to octave; furthermore, he says that he can find no reason for this asymmetry, leaving it to the discretion of the student whether to use the progression. Rather than articulating a strong prohibition against the “battuta,” he explicitly notes that the traditional rule is “of little importance.”

(a)

Forbidden by Mazzola and Fux	
Tritone Motion	Parallel Fifths
(0, 0)→(6, 6)	(0, 7)→(1, 8)
(0, 3)→(0, 9)	(0, 7)→(2, 9)
(0, 3)→(6, 3)	(0, 7)→(3, 10)
(0, 4)→(2, 10)	(0, 7)→(4, 11)
(0, 4)→(6, 6)	(0, 7)→(5, 0)
(0, 4)→(6, 2)	(0, 7)→(7, 2)
(0, 4)→(10, 10)	(0, 7)→(8, 3)
(0, 9)→(0, 3)	(0, 7)→(9, 4)
(0, 9)→(6, 9)	(0, 7)→(10, 5)
	(0, 7)→(11, 6)

(b)

Forbidden by Mazzola but not by Fux			
Repetitions	Exchanges	Dim. Triad	???
(0, 0)→(0, 0)	(0, 3)→(3, 0)	(0, 3)→(3, 6)	(0, 4)→(0, 0)
(0, 3)→(0, 3)	(0, 4)→(4, 0)	(0, 3)→(9, 0)	(0, 4)→(2, 2)
(0, 4)→(0, 4)	(0, 9)→(9, 0)	(0, 9)→(3, 0)	(0, 4)→(4, 4)
(0, 7)→(0, 7)		(0, 9)→(9, 6)	(0, 4)→(2, 6)
(0, 8)→(0, 8)			(0, 4)→(10, 2)
(0, 9)→(0, 9)			

Fig. 3. (a) Diatonic progression classes forbidden by both Mazzola and Fux. These include 9 of the 45 classes containing a tritone motion in one voice, and all eleven examples of parallel fifths. (b) Progression classes forbidden by Mazzola but not Fux. These include repetitions, exchanges in which two voices swap notes, four progressions outlining a diminished triad, and five unproblematic progressions that begin with a major third. Note that in all of these progressions, the cantus is listed first.

(a) (b) (c) (d) (e)

(0, 4)→(4, 0) (0, 4)→(4, 4) (0, 4)→(4, 0) (0, 4)→(10, 2) (0, 4)→(2, 6)

Fig. 4. Mazzola's forbidden progressions in Fux, Lassus, and Palestrina. (a–c) are from Fux 1971, 39–40; (d) is from Lassus *Cantiones duarum vocum* 2-5, m. 28; (e) is from Palestrina, *Missa Lauda Sion*, Gloria. In some of these examples, the cantus appears above the counterpoint, but Fux gives no indication that this is at all relevant to the progressions' acceptability.

3 Mazzola's Derivation of the Rules

So far, we have confined our discussion to the Fux rules themselves: accepting Mazzola's decision to work with pitch-class intervals, we have found the 65 progressions in Figure 2, rather than the 54 progressions mentioned in Mazzola's text. I now want to turn to Mazzola's *explanation* of these rules, focusing less on the mathematical details of his model than on its actual output.¹⁵ Mazzola argues from the fact that traditional counterpoint enjoins a careful balance between perfect and imperfect consonances, to the conclusion that consonances cannot progress completely freely. Instead, he claims there are "dissonances within consonances":

The first elementary rule of counterpoint "note-against-note" says that we are not allowed to take other intervals than the consonances [...] This seems evident, but it imposes a strong obstruction against another, more hidden directive: the idea of creating a tension between each interval and its successor. More precisely, the meaning of "contra" is not only that of a vertical opposition between cantus firmus and discantus. As Sachs has remarked [...], the preposition "contra" equally means a horizontal opposition between successive intervals in the given sequence. This requirement is not very explicit, but it is reflected in the distinction between perfect consonances (prime, fifth, octave) and the others, the imperfect sixths and thirds, and the idea of changing between perfect and imperfect consonances in order to create tension. This conceptual distinction seems to evoke a dissonant ingredient in the consonant character, although it does not really abolish the consonance, it is a kind of coloring effect.¹⁶

Mazzola proposes to use symmetry to capture this "dissonant ingredient," arguing (roughly speaking) that a progression between consonances $A \rightarrow B$ is allowable only if there is some affine transformation that sends A into a consonance and B into a dissonance (or vice versa).¹⁷

Having taught and studied counterpoint for some years, I admit that I was initially quite suspicious of this idea, largely because I could not imagine a plausible mechanism that would serve to connect Mazzola's mathematics to real-world musical practice. In particular, Mazzola's explanation—like many other contemporary accounts of diatonic music—somewhat anachronistically assumed a diatonic collection embedded within a twelve-tone chromatic scale; whereas in the Renaissance just intonation and even nineteen-tone equal temperament were active subjects of exploration.¹⁸ Furthermore, the form of Mazzola's argument seemed to involve a subtle logical leap, beginning with the indisputable fact that counterpoint requires blending perfect and imperfect intervals, but shifting to a mathematical model that provides *no guidance* about how to move between these two kinds of consonance. Instead the model delivers a whole collection of rules (avoiding parallel

¹⁵ For a readable introduction to the model, see Agustín-Aquino 2009.

¹⁶ Mazzola 2002, 646.

¹⁷ Again, for the details see Mazzola 2002 or Agustín-Aquino 2009. Here I am mainly concerned with the model's claims, rather the technical means by which they are derived.

¹⁸ See, e.g., Vincentino 1555.

fifths, tritone melodic intervals, and note repetitions) that have nothing to do with the motivating idea of “creating tension” by mixing types of consonance.¹⁹

Finally, and most importantly, Mazzola proposes a single explanation for three very different musical practices. The prohibition on tritone melodic motion has typically been explained, at least in part, by reference to the fact that tritones are *very difficult to sing*.²⁰ (On this view, the prohibition on tritones is analogous to the prohibition on major sevenths.) Parallel perfect intervals, by contrast, are quite singable; this prohibition is instead explained by the supposition that parallel perfect intervals lead to auditory fusion, thus decreasing voice independence.²¹ (This fusion, in turn, has been explained both by the acoustic properties of fifths and octaves, and also by the structure of the diatonic scale, which ensures that parallel diatonic octaves are always parallel chromatic octaves, while ensuring parallel diatonic fifths are mostly parallel chromatic fifths.²²) Mazzola’s counterpoint rules also include a host of non-Fuxian prohibitions on note-repetition—prohibitions that can only be justified by reference the rhythmic peculiarities of first-species writing. Intuitively, it seemed highly unlikely that these three independent considerations would come together such that they could all be modeled by interesting mathematical machinery. Were this to happen, it would be rather as if a bunch of pottery shards had spontaneously leapt up to form a beautiful unbroken vase.

To be sure, these are all defeasible worries: if it were true that Mazzola’s model could exactly reproduce Fux’s rules—even as understood by later theorists—then this would provide evidence that my initial suspicions had been wrong. If, on the other hand, there was only an approximate match between Mazzola and Fux, then this would tend to confirm my intuitions about the model’s irrelevance. And here it must be said that the results are not very good: of my sixty-five prohibited Fuxian motions, Mazzola prohibits only nineteen, or less than a third; at the same time, roughly half of his model’s prohibitions are acceptable according to Fux. (Figure 4 provides a few examples of these progressions, drawn from Fux, Lassus, and Palestrina.) This is on the face of it a strikingly bad fit between model and reality. A scientist with no prior bias in favor of Mazzola’s hypothesis would no doubt discard it for this reason.

There is also the important fact that Mazzola’s counterpoint rules are *not symmetrical under exchange of voices*. Thus we cannot answer the question “Is it permissible for one voice to move from F to A, while the other moves from A to F?” For if $F \rightarrow A$ is the cantus, then we have a forbidden $(0, 4) \rightarrow (4, 0)$ progression, while if $A \rightarrow F$ is the cantus, then we have a permissible $(0, 8) \rightarrow (8, 0)$ (cf. Figure 3, where only the first appears). I am aware of nothing in the contrapuntal literature that supports such a sharp distinction between cantus and counterpoint: on the contrary, the goal of contrapuntal composition is to create a collection of harmonious and

¹⁹ The model encodes the idea of “moving from consonance to dissonance and back” at a very abstract mathematical level, but these abstractions are connected only by analogy to the manifest practice of mixing perfect and imperfect consonances. Thus, Mazzola’s argument seems to require a prior conviction that there should be a metaphorical similarity between underlying mathematical relationships and the seemingly unrelated surface procedures.

²⁰ See Fux 1971, 35 (“hard to sing and sounds bad”).

²¹ This is precisely what allows organ builders to include a stop that adds a perfect fifth to the basic pitch.

²² See Tymoczko 2011, §6.3.

largely equal voices, with the significance of the cantus being simply that it is written first. Ironically, then, Mazzola ends up *breaking* one of the style's manifest symmetries (permutation of labels "cantus" and "couterpoint") in his attempt to construct a model that uses symmetry at a deeper mathematical level. This has the additional consequence of making it hard to falsify Mazzola's theory, since in actual music it is not always obvious which voice, if any, is "the cantus firmus."

Considerations of symmetry lead toward some fascinating methodological issues, which we unfortunately cannot pursue in full detail. In *Topos*, Mazzola explicitly rejects an approach in which intervals (or more generally, harmonies) are modeled as unordered sets:

"Our requirements would not be satisfied if we thought of an interval as being a set of pitch events. This would in particular not do justice to the concept of voices. The cantus firmus could not be distinguished from the discantus, and the crossing of voices could not be conceived."²³

Recent developments suggest that these claims are overstated: voice leadings (which represent *how the voices move the notes of one chord to those of another*) can be represented as continuous paths in the orbifolds representing unordered sets, just as voice crossings can be conceived as certain kinds of *paths that move through certain singularities*.²⁴ The Appendix shows that it is possible to interpret Fux's counterpoint rules as demarcating a set of "allowable moves" on the orbifold representing unordered pairs of pitch classes. What results is a plausible model of counterpoint in which voices are equal partners, rather than a hierarchical conception in which the discantus is a mere "melodic variation" on the cantus line.²⁵

This new model, however, is a matter for another paper. Let us instead return to the statistical argument Mazzola uses to justify his view. He notes "if somebody tries to hit at least 21 of the 54 inadmissible cases of the reduced strict style without knowing anything about counterpoint by 37 trials, the chance is less than $2 \cdot 10^{-8}$." This is not convincing, since there are countless mathematical formulae which, for purely coincidental reasons, correspond in surprising ways to real-world facts. (For example, there is an urban legend according to which U.S. presidents die in office, usually by assassination, when elected in years evenly divisible by 20.²⁶) Mazzola's style of argument, if valid, could be applied equally well to any of these cases. (It is, for instance, unlikely that a random sequence of numbers would reproduce the election years of presidents who die in office.) The specific problem here is that the

²³ Mazzola 2002, 619.

²⁴ See Tymoczko 2006, Callender, Quinn, and Tymoczko 2008, Tymoczko 2011, and Hall and Tymoczko n.d. Hall and Tymoczko explicitly note that voice crossing in the singular space $\mathbb{R}^n/\mathcal{S}_n$ is represented by a line segment that "bounces off" the singularity.

²⁵ Mazzola 2002, 617.

²⁶ Harrison, elected in 1840, died of pneumonia; Lincoln, elected in 1860, was assassinated in 1865; Garfield, elected in 1880, was assassinated in 1881; McKinley, re-elected in 1900, was assassinated in 1901; Harding, elected in 1920, died of a stroke in 1923, and was said to have been poisoned; Roosevelt, re-elected in 1940, died of a stroke in 1945; Kennedy, elected in 1960, was assassinated in 1963; and Ronald Reagan, elected in 1980, was shot and nearly killed. Proponents of the theory sometimes argue that Reagan's survival broke the "curse."

predictions of Mazzola's theory are not independent trials, analogous to throwing a series of darts at a dartboard: Mazzola gives us a *single* theory which makes a *single* inseparable set of claims about which motions are permissible. If there is an experiment being performed, it is the endlessly repeated *Human Being Wonders Whether There is Some Connection Between Mathematical Fact and Real-World Phenomenon*. Thus, rather than asking "what are the odds of getting 21 hits in 37 trials?" we should be asking "if we engage in something like 10^8 experiments with 25–50 trials in each, what are the odds that at least one of them will contain 21 successes?"²⁷ These odds are of course quite good, which is precisely why we require that mathematical explanations of real-world phenomena either be extraordinarily accurate, or else be accompanied by some plausible explanatory mechanism. Mazzola's theory, as we have seen, is problematic on both counts.

Finally, it is important to realize that standards of accuracy depend crucially on context. In some cases, it could be useful to have a test that was 30% accurate while delivering a 50% rate of false positives.²⁸ The situation we are considering is not of this sort, largely because we *already have* a fairly satisfactory derivation of the fundamental Fuxian prohibitions: tritones are dissonant and difficult to sing; parallel perfect intervals are more likely to fuse, for reasons of both acoustics and scale structure; and note repetition at slow tempi creates a kind of inhomogeneity that is at odds with the contrapuntal aesthetic. From this standpoint, Mazzola's explanation has to pass a relatively high crossbar: the crucial question is not whether it is better than *no explanation at all*, but whether it is better than the very good explanations we already have.

4 Conclusion

It might seem unfair to draw any conclusions about a 1300-page book on the basis of a single small section of a single chapter. But for some theorists, the value of models lies precisely in their real-world applications: do they, or do they not, teach us something new about music as has actually been practiced? These readers will be prepared to accede to *Topos*'s extraordinary demands only insofar as they receive an extraordinary payoff in return. And in the single case we have examined, the payoff is anything but remarkable: we have found significant problems at almost every stage of Mazzola's argument, from the description of Fux's rules, to the accuracy of the model, to the baffling use of statistical arguments. Given the severity of these issues, some readers may be tempted to draw negative conclusions about *Topos* itself.

²⁷ The number 10^8 is made-up. Nobody knows what the actual number is, though we can be certain that it is large.

²⁸ Suppose there were some terminal disease that only struck middle-aged people, but which could be prevented by giving infants an expensive drug with no side effects: if a test could capture 30% of those who would be afflicted, and if there were no adverse health effects to administering the drug to healthy babies, then discovering such a test would be a tremendous achievement that saved many lives.

Sometimes, problems with a part really do cast doubt upon the whole.²⁹

Other readers will not feel so pessimistic. Confronted with a mismatch between theory and the world, we can always respond by saying “well, so much the worse for the world.” And one can legitimately argue that Mazzola’s system has a beauty all its own, a compelling logic that overrides the occasionally tenuous assertion about reality. One possibility is to treat *Topos* as a radical and challenging artwork that adopts the guise of scholarship, but whose ultimate goal is beauty rather than truth. Another is to allow *Topos* the sort of autonomy we grant to abstract mathematics: for just as we do not criticize higher-dimensional hyperbolic geometry for being inapplicable to the ordinary 3D world, so too can we declare that Mazzola is engaging in a kind of “theoretical music theory”—an activity that is more concerned with how we *might* think about music than with solving particular historical or analytical problems. Perhaps *Topos* is a kind of mathematics whose questions originate in familiar musical problems, but have managed to transcend them.³⁰

For me, these are the fundamental issues: Is music theory an applied science, whose value is largely determined by its practical results? Or is it something self-standing, whose worth lies in the intrinsic beauty of its constructions? How do we evaluate a work like *Topos*, which is trying not just to describe previous music, but also to open up new conceptual territory? How much do we prioritize mathematical sophistication relative to musical insight? When doing music theory, should we aspire to the disinterested caution of the scientist, or the committed intensity of the visionary modern artist? And what role should philosophy play in the enterprise? These questions, ultimately, are not matters for intellectual debate, but rather opportunities to display our fundamental intellectual values. Different answers are hard to reconcile, precisely because there is little neutral ground on which impartial discussion can take place.

From this point of view, the very inadequacies of *Topos*, when viewed under the lens of practical, results-oriented science, strongly suggest that Mazzola is playing a different game altogether.³¹ There are, indeed, many activities that go under the rubric of “music theory,” and it may be that Mazzola and I are quite far apart in our conceptions of the discipline. If so, my criticisms should be taken as an invitation for two music-theoretical traditions—largely but not exclusively associated with Europe

²⁹ Roeder 1993 critiques two other applications that appear throughout Mazzola’s writing: his model of “cadences” and his reading of keys in the “crisis” passage in Beethoven’s *Hammerklavier*. This leads him to declare that Mazzola’s theory “is impressive but not viable.”

³⁰ It might also be reasonable to treat *Topos* as something like an autobiography. Guerino is, after all, an extraordinary individual: a jovial and genuinely talented mathematician, a gifted free-jazz pianist, an a European pioneer of mathematically informed music theory. *Topos* may perhaps be read as the record of a unique and extraordinary man’s attempt to grapple with the mystery of music. From this point of view, there is something charming about the generous manner in which the book incorporates the work of Guerino’s students and collaborators. A charitable lack of skepticism can sometimes be a virtue in a teacher.

³¹ As Roeder writes: “Americans value music theories for their practical (analytical or compositional) application, and usually present them with little philosophical motivation. In contrast, Mazzola values his theory primarily according to how well it realizes a particular philosophy about the nature of music” (1993, 307).

and America—to engage in a deeper and more sustained philosophical dialogue about fundamental goals and methods.³²

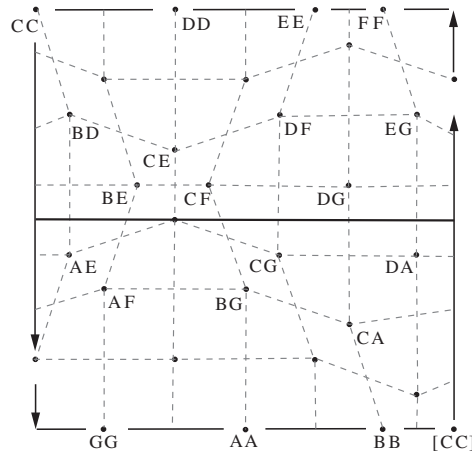


Fig. A1. The orbifold $\mathbb{T}^2/\mathcal{S}_2$, with only diatonic consonances labeled. Unlabeled points correspond to diatonic dissonances, while dashed lines represent stepwise motion within the diatonic scale. The left edge is glued to the right with a half twist. The top and bottom boundaries act like mirrors, while the dark line at the center of the space contains tritones.

Appendix

Figure A1 shows the orbifold $\mathbb{T}^2/\mathcal{S}_2$, representing unordered pairs of pitch classes. The space is a Möbius strip whose singular boundary acts like a mirror. Though the space is continuous, only diatonic consonances are labeled. Voice leadings are represented by the images of line segments in the covering space \mathbb{R}^2 , “generalized line segments” that may bounce off the singular boundary in the quotient. We can think of the space as a kind of gameboard on which composers of two-voice music necessarily ply their trade. Fux’s first-species rules determine a specific game that can be played on the board; an alternate game, modeling 11th-century contrapuntal practice, is discussed in Chapter 6 of Tymoczko 2011. For simplicity, I will model Fux’s rules in the case where one is composing both voices at once. Working with a precomposed cantus amounts to fixing one 45° diagonal component of each generalized line segment.

³² These issues are reminiscent of those separating “Anglo-American” and continental approaches to philosophy. Anglo-American philosophers typically prize conceptual clarity and rational argument, conceiving of their enterprise as being continuous with science, and ignoring philosophers such as Hegel and Schopenhauer. By contrast, continental philosophers often ask broader questions, conceive of their work in (partially) aesthetic terms, and take philosophers like Hegel quite seriously. Here, as in music theory, it is difficult to bridge this gap precisely because the dispute is (in part) about what sorts of intellectual inquiry are worth pursuing.

Note that the following rules do not aspire to have the explanatory force of Mazzola’s model. Instead, they consider traditional rules to be postulates that are explained outside the theory (e.g. by considerations such as those in §3). The goal is to show that, contra Mazzola, one *can* represent Fux’s counterpoint rules using the geometry of unordered sets, and that geometry provides an intuitive characterization of the possibilities available to composers—one in which the musically salient alternatives (involving efficient voice leading) are also *geometrically* salient (being represented by short paths on the orbifold). For more on the underlying philosophy, see Tymoczko 2011.

Rule 1 (*voice leadings*). Each move occurs along “generalized line segments” as defined above.

Rule 2 (*consonant intervals*). A move can begin and end only at the labeled points (representing octaves, thirds, sixths, and fifths). In the case of octaves, one must “desingularize” by beginning and ending infinitesimally close to the boundary, rather than exactly on it.

Rule 3 (*no tritones*). One cannot move along any path whose projection onto either 45° diagonal is six units long.³³

Rule 4, v.1 (*no strictly parallel fifths or octaves*). One cannot move *horizontally* from one fifth to another, or from one octave/unison to another.

Rule 4, v. 2 (*no parallel or antiparallel fifths or octaves*). One cannot move *in any way* from one octave to another, or from one fifth to another.

Rule 4, v. 3 (*no similar motion into perfect consonances*). One cannot move into an octave or fifth along a generalized line segment whose (initial) slope is in the range $(-1, 1)$.

Rule 5 (*no fourths, full version*). At the very beginning of the game, when one first chooses a point on the Möbius strip, one must choose an integer, the *tritone parameter*, representing the integer part of the distance between voices, measured in tritone units from lower note to higher note.

This number is incremented or decremented each time a generalized line segment either (a) bounces off the mirror boundary; or (b) crosses the horizontal line of tritones at the center of the strip. When the number is even, decrement when crossing the line of unisons and increment when crossing the line of tritones; when the number is odd, increment when crossing the line of unisons and decrement when crossing the line of tritones. When decrementing from zero, move to “-0” before moving to -1. Similarly, when incrementing from -1, move to -0 before 0.

One can arrive at a point representing fourths and fifths only if the tritone parameter will at that time be odd (counting both 0 and -0 as “even”).

Together these rules demarcate the complete set of Fux-admissible paths in $\mathbb{T}^2/\mathcal{S}_2$. That is, any geometrical motion in accordance with the five rules can always be realized in pitch space (“lifted”) so that the resulting progression conforms to Fux’s rules; conversely *every* Fux-admissible passage of first-species counterpoint (in the sense of Figure 2) projects from pitch-space into $\mathbb{T}^2/\mathcal{S}_2$ as a series of motions in

³³ More strictly: one cannot move along any generalized line segment that is the image, in the covering space, of a path either of whose diagonal projections are 6 units long.

accordance with the rules. These Fux-admissible paths can be sorted into three classes: *invertible paths* which contain no fifths, so that every pitch-space realization (“lifting”) conforms to Fux’s rules; *mobile counterpoint*, in which the two voices can begin at several different distances from one another (e.g. for one voice starting on C4, the other can start on G4 or any higher G); and *immobile counterpoint*, in which the starting distance between voices is completely determined.

A few final points. First, voice-crossings occur when the tritone parameter changes sign. Second, the three versions of Rule 4 show that we can represent a range of prohibitions on motion into perfect consonances, including the rule actually articulated by Fux. And third, there are alternatives to Rule 5 that eliminate some but not all octave information.³⁴

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³⁴ For instance: “once you have begun the game, you can move freely at any time to any perfect fourth/fifth; however, having done so you must set the tritone parameter to an odd number; henceforth you can arrive at a fourth/fifth only if the tritone parameter will be odd [forgetting about that –0 business].” This rule permits all Fuxian sequences that can be realized without voice crossings.