Chapter X

Intuitive Musical Homotopy

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Voice leading is closely connected with homotopy, the exploration of paths in higher-dimensional configuration spaces. Musicians explored these spaces centuries before mathematicians developed tools for describing them. In this paper we analyze the group structure of these contrapuntal paths, generalize traditional music-theoretical vocabulary for representing voice leading, and ask which paths are realizable given our generalized vocabulary.

1. Introduction

Intuitively, a *voice leading* is a way of moving from one chord to another. Formally, it can be represented as a vector or path in the configuration space of possible chords (Callender, Quinn, and Tymoczko 2008, Tymoczko 2016). The collection of all bijective voice leadings from a chord to itself is the *orbifold fundamental group*, a generalization of the fundamental group to singular quotient spaces (Hughes 2015). This can be extended to the *voice-leading group*, modeling the bijective voice leadings linking transpositionally related chords within some scale (or, in the limit, continuous unquantized chord space); a further extension allows us to consider nonbijective voice leadings in which one or more notes are doubled. In each case, voice leading is closely analogous to homotopy, an exploration of loops and paths in an abstract configuration space. Contemporary mathematics provides tools for describing these loops, allowing us to understand the contrapuntal routes from a chord to itself, or from one chord to another.

For centuries keyboard players have employed simple physical heuristics to accomplish a similar purpose. David Heinichen, in his *Generalbass,* recommended that improvising keyboardists keep their right hand in "close position" when playing triads (Figure 1, Buelow 1992). These close-position voice leadings comprise an important subgroup of the three-note voice-leading group, containing voice leadings with *no crossings in pitch-class space* (Tymoczko 2008, 2011). (Note that in this context, a keyboard player cannot physically differentiate between registrally equivalent arrangements of voices, such as those in Figure 2; this limitation that will remain in force throughout this paper.) This subgroup is the quotient of the voice-leading group by the *voice exchanges,* a normal subgroup generated by those voice leadings that swap two adjacent pitch classes along equal and opposite paths (Figure 3). Thus musicians long ago identified simple, physically realizable gestures that generate a mathematically interesting subset of contrapuntal possibilities—namely those without voice crossings in pitch class space.

In this paper we develop this connection between the practical and the theoretical, asking how musicians might understand and manipulate the large number of voice leadings theoretically available to them. We begin by analyzing the structure of the voice-leading group. We then generalize the notion of "close position" to five physical configurations—registral arrangements of upper voices, all musically familiar and common in many contrapuntal genres. We then ask "what portion of the theoretically possible voice-leadings are reachable using these configurations?" Our main result is that these generalized configurations allow composers to reach a substantial proportion of the "nearby" possibilities, including all of those with less than four voice crossings in pitch-class space.

2. The voice leading group and its extensions.

A *path in pitch class space* is an ordered pair whose first element is a pitch class (point on a circle) and whose second is a real number (element of the tangent space at that point) representing how the note moves (Tymoczko 2011, 2016). In this context there is an isomorphism between elements of the tangent space and homotopy classes of paths in the circle, so we can conceive of paths in pitch-class space as motions in the circle. A *voice leading* is a multiset of paths in pitch-class space, representing motion from one chord to another. Voice leadings can be transposed and inverted in the obvious way: transposition acts on a voice leading's pitch classes while leaving its real numbers unchanged; inversion inverts the pitch classes while multiplying the real numbers by –1 (Tymoczko 2011).

The collection of bijective voice leadings from a chord to itself form a group, the orbifold fundamental group for the configuration space of n-note chords, $\mathbb{T}^n/\mathcal{S}_n$. This can be written as $\mathcal{S}_n \ltimes \mathbb{Z}^n$, with $\mathcal{S}_n$ factor permuting the chord's notes and the $\mathbb{Z}^n$ factor displacing them by octaves (Hughes 2015). We will find it useful to rewrite this as $\mathbb{Z} \ltimes (\mathcal{S}_n \ltimes \mathbb{Z}^{n-1})$, with $\mathbb{Z}$ the *crossing-free subgroup* and $(\mathcal{S}_n \ltimes \mathbb{Z}^{n-1})$ the subgroup of voice exchanges. As mentioned, the voice exchanges are generated by the n pairwise swaps that exchange adjacent notes in pitch-class space; together, they span an n–1 dimensional subspace of the configuration space of possible chords (Callender, Quinn, and Tymoczko 2008). For a nondegenerate chord (i.e. chord whose pitch classes are all distinct), the voice-exchange subgroup is independent of the structure of the chord or the scale it is in; it depends only on the chord's cardinality. The voice exchanges contain all and only those bijective voice leadings $X \rightarrow X$ whose paths sum to 0, while the paths in the crossing-free subgroup sum to an integral number of octaves. From this it follows that $xyx^{-1} \in (\mathcal{S}_n \ltimes \mathbb{Z}^{n-1})$ for $x \in \mathbb{Z}$ and $y \in (\mathcal{S}_n \ltimes \mathbb{Z}^{n-1})$, and hence that $\mathcal{S}_n \ltimes \mathbb{Z}^{n-1}$ is normal. This mathematical result underwrites the common musical practice of ignoring or "factoring out" voice crossings and voice exchanges (Tymoczko 2011, Forte and Gilbert 1982).

The *voice-leading group* connects transpositionally equivalent chords, with voice leadings acting in the natural way: if $\mathcal{V}(a)$ is any voice leading from a to one of its transpositions, then $\mathcal{V}(\mathbf{T}_n(a))$ is defined as $\mathbf{T}_n(\mathcal{V}(a))$ (Tymoczko 2011), where **T** is the transposition operator. We can represent these voice leadings as abstract schemas such as "keep the root of the major triad fixed, move its third up by semitone, and its fifth up by two semitones," descriptions that will be unambiguous so long as the chord is not transpositionally symmetrical and contains no duplicate pitch classes. We can write the voice-leading group as $(\mathbb{Z} \times \mathcal{C}_{gcd(c,\,n)}) \ltimes (\mathcal{S}_n \ltimes \mathbb{Z}^{n-1})$, with $\mathcal{C}$ the cyclic group and $gcd(c, n)$ the greatest common divisor of the size of the chord (n) and the size of the enclosing scale (c).[a] This decomposition can be represented graphically as shown in Figure 4, with a line winding n times around the circular dimension of an annulus, the c transpositions of the chord equally spaced along it. Chords sharing the same angular coordinate have pitch classes summing to the same value. The counterintuitive n-fold winding is a manifestation of the chord's n inversions, which are equivalent only assuming octave equivalence; in continuous space one can move orthogonally to the line of transposition between these n inversions, along a series of paths in pitch class space summing to 0.

Crossing-free voice leadings are represented by homotopy classes of paths in the annulus. To understand their structure it is useful to adopt the conception of a scale as a metric, described in Chapter 4 of Tymoczko 2011; this allows us to use scale-degree numbers even in continuous pitch-class space. Angular motion along the line corresponds to transposition along the scale, with clockwise motion $1/x$ of the way around the circle transposing downward by $1/nx$ steps, and radial motion between adjacent lines representing the "diagonal action" that transposes the notes of the chord up $1/n$ of an octave while transposing the notes of the chord down one chordal step, thus leaving the sum of its pitch classes unchanged. The resulting chord will not in general lie in the same scale. However, by modeling scales as metrics we can conceive of these chords as points in continuous space, labeled with scale-degree numbers.

The subgroup $\mathcal{C}_{gcd(c,\,n)}$ contains all the crossing-free voice leadings whose paths sum to zero; it is the only finite, crossing-free

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[a] For continuous space this becomes $\mathbb{R} \times \mathbb{C}_n$.

subgroup of the voice-leading group.[b] When n divides c there are n distinct copies of the chord at each radial position, related by transposition $1/n$, as in Figure 4a; here the subgroup $\mathcal{C}_{gcd(c,\ n)}$ is the quotient of the crossing-free subgroup by transpositional voice leadings. When n and c share a common factor, there are $gcd(c, n)$ copies of the chord sharing the same pitch-class sum, as shown in Figure 4b; here we quotient out just some of the scale's transpositions. And when the sizes of chord and scale are relatively prime, the crossing-free subgroup is just $\mathbb{Z}$, whose generator is Hook's "signature transform" (Figure 4c, Hook 2008, 2013, Tymoczko 2013). In this case, the voice-leading group is isomorphic to the orbifold fundamental group, even though it relates a larger collection of transpositionally related chords. Figure 5 shows a variety of graphs for chords and scales of various sizes, all clearly similar.

The annular structure of all of these graphs reflects the topology of the underlying orbifold $\mathbb{T}^n/\mathcal{S}_n$, whose nonsingular interior is the twisted product of a n–1 simplex with a circle. Angular motion on our graphs corresponds to motion along the circular dimension of $\mathbb{T}^n/\mathcal{S}_n$. A voice leading that makes a complete circle in this dimension acts as a "transposition along the chord," moving voices by the same number of steps along the chord itself—for example, from C up to E, from E up to G, and from G up to C in the major chord {C, E, G}. The radial dimension of the annular graph collapses the n–1 dimensions of the simplex; consequently, a sequence of purely radial motions $x_1 \rightarrow x_2 \rightarrow \ldots \rightarrow x_1$ will leave all voices where they began. Radial paths are therefore *contractible* whereas complete angular circles are not—as we can see both on the annular graphs and by the topology of the orbifolds they model.

A characteristic difference between the theory of voice leading and the earlier "transformational theory" is that the former emphasizes the nontrivial effect of these noncontractible circles—representing nontrivial voice leadings from a chord to itself. Traditional

[b] It is clear that any finite subgroup must have paths summing to zero, since we can iterate any voice leading with sum x to obtain an infinite collection of voice leadings whose sums are ix, for $i \in \mathbb{Z}$.

"transformational" models sometimes incorporate these facts in a tacit or unrecognized way. For instance, an "LP cycle" on the familiar Tonnetz leaves voices unchanged, whereas a "PR cycle" acts as a one-step scalar transposition; this difference is evident when Tonnetz-motions are translated into music, even while remaining obscure in the geometry of the Tonnetz itself. (See Figure 6 and Cohn 1997, pp. 35–36). The theory of voice leadings helps explain these puzzles: the LP cycle links C, E, and A♭ major triads, forming a contractible circle in the orbifold $\mathbb{T}^3/\mathcal{S}_3$, and represented solely by radial motion on Figure 5a, whereas the PR cycle links C, E♭, G♭, and A major triads, traversing a noncontractible circle in chord space—represented on Figure 5a by a full clockwise turn. This distinction is obscured by the common belief that the Tonnetz is a torus with two distinct noncontractible axes; cf. Tymoczko 2012. Many other transformational models are similar to the Tonnetz in implicitly encoding the structural relationships that emerge explicitly in the geometrical approach—details that, in the earlier theory, sometimes become evident only when we translate the model's abstract geometrical motions into actual musical notation.

We can extend the voice-leading group to include non-bijective voice-leadings, or voice leadings with doublings. For instance, consider the group of four-voice voice leadings that connect triads in the C major scale, with one note of each chord doubled. Here there are three possible doublings corresponding to the three notes of the chord: for each particular doubling we have a subgroup of the four-note voice-leading group $\mathbb{Z} \ltimes (\mathcal{S}_4 \times \mathbb{Z}^3)$, equivalent to Figure 5a.[c] To these possibilities we need to add the voice leadings that move the doubling from one chordal element to another. A natural choice here are the zero-sum, crossing-free voice leadings that move the doubling from root to third to fifth and back (Figure 7). This gives us the *four-voice triadic voice-leading group* $(\mathbb{Z} \times \mathcal{C}_3) \ltimes (\mathcal{S}_4 \times \mathbb{Z}^3)$. Extending this construction to arbitrary chords in arbitrary scales is an interesting problem which we leave for the future, as it is not relevant to what follows.

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[c] This is a subgroup because of the degenerate voice leadings that swap identical pitch classes.

3. Five registral configurations

We are interested in the connection between simple physical heuristics, such as "keep the chords in close position," and the voice leading group. Specifically, we ask "what proportion of the theoretically possible voice leadings are *performable* with a given set of physical gestures?" We are not, for the purposes of this paper, interested in forbidden parallels or the avoidance of second-inversion triads: our only question is which voice leadings can be realized given a set of physical options. (The acceptability of those voice leadings, according to the tradition, is a separate question.) Furthermore, in keeping with Figure 3, our notion of "performability" is keyboard-based, forbidding crossings between registrally indistinguishable chords.

For any chord in any scale, close-position voicings suffice to generate all the crossing-free bijective voice leadings between nondegenerate chords of any cardinality. We can extend this result, in the triadic case, by adding the well-known "open" position, which separates the notes of triad so there are two triadic steps between registrally adjacent notes (Figure 8). Moving from close to open position involves one pairwise voice exchange; furthermore, any pairwise voice exchange can be represented in this way by placing the relevant notes in the top and bottom voice. Thus, the second physical configuration expands our contrapuntal range: a composer who uses only close and open positions can access all and only those bijective, triadic voice leadings with at most one crossing.

We now describe a collection of possible configurations for the upper voices in four-voice triadic counterpoint, in which every sonority is a complete and exactly two voices sound the same pitch class. If we require that voices be separated by an octave or less, then there are just the five possibilities in Figure 9: "doubled interval," where two voices sound the same note and the third less than an octave away; "close position" where the voices sound three different pitches spanning less than an octave; "half-open," where the outer voices are octave apart and the middle voice is between them; "open position," where the voices sound a complete triad spanning more than an octave but with adjacent notes separated by less than an octave; and "open octave," where two adjacent voices are an octave apart, the third less than an octave away from its nearest neighbor. These are arranged here in order of increasing separation between voices, so that the "half-open" can genuinely be said to lie between close and open. Clearly, there are two categories here, the complete triadic voicings (C and O) and those with doublings (DI, H, OO). Figure 10 shows that these are related by octave displacements: we can turn DI into H, C into O, and H into OO, by transposing the middle voice up by octave. Since we require that all triads be complete, the pitch class of the fourth voice is determined by the content of the upper voices in DI, H, and OO position; in C or O position, the bass voice can sound any tone.

Figure 11 shows the distribution of upper-voice configurations in Bach's chorales and the four-voice passages of Palestrina's masses. Palestrina's upper voices are generally close together, concentrated toward the left side of the graph, while Bach features the C↔H↔O subsystem. (This may reflect the greater importance of voice-crossings in Palestrina's style, and their almost total absence from the chorales.) Figure 12 shows the most popular triadic transitions in the chorales: almost a third of the voice leadings connect close-position triads, with another 10% connecting open position to open position; the seven next-most popular patterns connect adjacent positions on the model. Note the frequency of direction motion between C and O, which is slightly more popular than motion to and from the open-octave position. These common motions are modeled by the lines on Figure 11.

Teachers can use these five positions to provide students with a simple and symmetrical set of contrapuntal guidelines: DI goes to C, C goes anywhere except OO, H goes either to C or O, O goes anywhere except DI, and OO goes to O.[d] Together, these rules cover about 85% of the four-voice triadic voice leadings in the Bach chorales and about 75% of those in Palestrina's masses, with Palestrina's greater use of pitch-space voice crossings (e.g. Figure 3) accounting for much of the difference. The approach simplifies the long lists of voice leadings sometimes found in textbooks (McHose 1947, Kostka and Payne 2003),

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[d] Note that H→H voice leadings will have parallel octaves unless the doubled notes stay fixed.

allowing students to produce idiomatic part-writing quickly and with a minimum of pain: the focus on upper-voice configurations helps them avoid parallels, encourages them to emphasize spacing, and gives them a *positive* set of options to think about. (They know, for example, that if they are in H position then they will be moving to either C or O on the next chord; furthermore they know that it is impossible for upper voices to form parallel octaves following these rules, and unlikely for them to form parallel fifths.[e]) It can also be useful to have students label the configurations in pre-existing music, so that they can see for themselves how they typically behave.

4. Performability of voice leadings

Suppose a keyboard player is generalizing traditional keyboard practices by using their right hand to articulate three voices in some subset of the DI, C, H, O, and OO, positions, with their left hand unconstrained. What voice leadings can be reached in this manner? That is, for a given set of upper-voice configurations, which voice leadings can and cannot be performed? In asking this question we will assume that the two hands can be separated so that their voices do not cross in pitch space, or perhaps played on separate manuals of a multi-manual instrument such as an organ; thus there is no question of whether the bass crosses upper voices. We also acknowledge that the performability of the OO voicings is somewhat unrealistic, requiring unusually large hands.

Our question fundamentally concerns the orbifold fundamental group, the collection of voice leadings from a chord to itself. For suppose we have a four-voice voice leading between two distinct triads such that its upper three voices are free of crossings while also connecting sonorities in the OUCH positions. First, we can transpose the destination chord so that it is equivalent to the starting chord (ignoring the doubling) without changing the position of its upper voices, as in Figure 13. Second, we can transform the destination chord so that its

[e] For example a DI↔C transition can produce fifths only if moving between a close, root position triad and a fifth with one note doubled; H↔O transitions can produce fifths only between adjacent voices.

doubling is equivalent to that of the first chord by transposing all notes along the triad, as in Figure 14; this will again preserve the arrangement of upper voices, sending DI to DI, C to C, H to H, O to O, and OO to OO. It follows that we can restrict our attention to voice leadings from a chord to itself with the very same doublings; in other words we consider the voice-leading group from the triadic multiset {C, C, E, G} to itself.

These manipulations indicate that we should not focus on the absolute distance moved by the voices, since that can be changed by adding an arbitrary transposition to the destination chord, but rather the number of *voice crossings in pitch-class space* contained by the voice leading. In pitch space, the number of voice crossings is equal to the number of crossed voices, but in pitch-class space two voices can cross multiple times: for instance, the voice (D, –14) crosses the voice (C, 1) twice. Geometrically, the number of voice crossings records the number of times the voice leading's path touches the singularities of the orbifold $\mathbb{T}^n/\mathcal{S}_n$. In the three-voice context, we have seen that close position by itself allows the expression of every voice leading with zero crossings, while close and open can express voice leadings with one or fewer crossings. More right hand positions will allow us to perform voice leadings with more crossings.

Before considering four-voice triadic case, it is worth looking at the simpler situation of four-note chords with no doublings, such as seventh chords. Since there are no doublings, the upper three voices can be in just two positions: close (spanning less than an octave) and open (spanning more than an octave but with less than an octave between adjacent notes). Figure 15 shows that all voice leadings with fewer than four crossings are performable, with the number of performable voice leadings reaching a maximum of 128. The total number of voice leadings with n crossings, when divided by four, yields the series 1, 4, 10, 20, 34, 52, 74, 100 …, or $2n^2 + 2$ for $n > 0$. This is the number of exterior points on the tetrahedra representing the tetrahedral numbers 1, 4, 10, 20, 35, 56, 84, 120, 165, 220; our sequence is equal to the nth tetrahedral number t_n for $n < 4$, and $t_n - t_{n-4}$ otherwise (Sloane's A005893; see Deza and Deza 2012, p. 126, OEIS Foundation 2017). The musical interpretation is as follows: crossing-free voice leadings whose paths sum to 0 are represented geometrically by voice leadings within a tetrahedral region that tiles three-dimensional space (the "cross section of four-note chord space" discussed in Tymoczko 2011), which contains four "modes" of each sonority, represented on Figure 4a by chords sharing the same radial position. The voice leadings with *n* crossings are found within the tetrahedral regions that can be reached from some particular region by exactly *n* face-preserving reflections. These tetrahedral regions themselves form the exterior of a tetrahedron isomorphic to that which is used to represent the *n*th tetrahedral number.[f]

Turning now to four-voice voice leadings between triads: if we require that the left hand stay in just the C position (or equivalently, just O), then we can obtain three of the four crossing-free voice leadings from a doubled triad to itself. The outlier, shown previously as the second voice leading in Figure 13, is what Tymoczko has called a *nonfactorizable* voice leading—that is a voice leading between four-voice triads that does not contain three voices articulating two complete triads (Tymoczko 2011, ch. 7). If the right hand can use *both* C and O, then we can obtain all the one-crossing voice leadings, but we still cannot reach Figure 13's nonfactorizable voice leading. (That voice leading, in other words, is the only nonfactorizable voice leading with at most one crossing.) If we allow, O, C, and H, then we can perform all the voice leadings with at most one crossing, but we cannot perform the two-crossing voice leadings in Figure 16. And if we permit all of the five configurations, then we can obtain all the voice leadings with *three* or fewer voice crossings. Figure 17 shows the total number of voice leadings for various numbers of crossings, and the percentage of these performable with various upper-voice configurations. Note that the doubling now obscures the connection to the tetrahedral numbers discussed earlier.

We conclude that relatively simple set of physical heuristics, all performable at the keyboard and all common in the musical literature, allow access to a substantial range of contrapuntal possibilities. More precisely, the five upper-voice configurations of Figure 11 provide a conceptual tool sufficient for manipulating the 58 four-voice triadic

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[f] Similarly, the three-voice voice leadings with *n* crossings correspond to the number exterior points on the triangle representing the *n*th triangular number.

voice-leading patterns with the fewest crossings, and about 80% of the 100 patterns with the next-fewest crossings. (Recall here that we are considering only voice leadings from a chord to itself with the very same doubling, as discussed in connection with Figure 14: by allowing the position of the doubling to vary, or allowing chords to be transposed, we obtain many more possibilities.) Note furthermore that these voice leadings can be performed without voice-crossings in *pitch* space, such as those in Figure 3; in this sense, crossings in pitch space are not necessary for exploring a wide-range of voice leadings. Our configurations are simple enough that they can be internalized by any musician, regardless of mathematical aptitude. These schemas thus provide a link between the embodied world of practical musicianship and the abstract space of topological possibility—showing us exactly what proportion of the homotopy group can be easily accessed by keyboardists.

Our approach treats the upper voices as a unit and the bass as an independent actor. An interesting question is whether this division is simply a matter of convenience or whether it grounded in identifiable features of musical practice. Historically, these ideas originate in figured-bass pedagogy, which described the C, O, and H configurations as convenient hand positions allowing keyboardists to generate four-voice counterpoint in real time; in these styles, the 3+1 division represents a genuine fact about the music, which treats of the bass as distinct from the chordal upper parts and assigns them to different hands. We have generalized the traditional categories of figured-bass pedagogy to encompass *all possible upper-voice configurations* in which the voices are registrally ordered and less than an octave apart. This generalized strategy is potentially applicable to any music that limits the distance between voices, whether conceived in a 3+1 fashion or not. It is therefore worth asking to what extent different musical repertoires support this partitioning into bass and upper voices, or whether it is merely an external framework that we have imposed on the music.

The answer is that the bass typically plays a exceptional role, though the degree of independence varies from repertoire to repertoire. In the four-voice chords found in Palestrina's masses, for example, the bass is most likely to sound chord roots (69% of all sonorities vs. 25–

30% for the other voices), most likely to move by fifth (9% of all intervals vs. 4–5% for the others), most likely to have its note doubled by another voice (32% vs. about 23%) and is separated on average by the greatest distance from the nearest voice (by about half a diatonic step more than the others). These subtle asymmetries are larger in other genres, not just later music such as Bach's chorales, but also the more chordal genres of the sixteenth century—Frottole, for example, or Goudimel's 1560s harmonizations of the Geneva Psalter. In this sense, it seems that the 3+1 division is not just a mathematical conceit, but a distinction supported by even the exemplars of Renaissance polyphony.[g] One might say that it points toward a subtle and implicit precursor to later chordal thinking.

Voice leading involves complicated mathematics, groups representing homotopy classes of paths in higher-dimensional, singular spaces. These spaces were explored by practical musicians long before mathematicians developed the concepts needed to describe them. Upper-voice configurations are an intuitive tool for manipulating these musical possibilities, learned implicitly and deployed not just by keyboard players but composers more generally. Modern music theory allows us to connect these two approaches, making explicit the musician's implicit knowledge, giving students new tools, and allowing us to appreciate the deep and inherently mathematical knowledge of the practical musician.

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[g] A further subtlety: in a genre that makes frequent use of voice crossings, there is a difference between *the lowest voice acting special* and *there being a single musical voice that acts like a special lowest part.* In Palestrina's music, for example, the tenor sometimes crosses below the bass and "acts like a bass voice" for a while.

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Fig. 1. In close position, voicings are ordered registrally and span less than an octave.

Fig. 2. Keyboard players find it difficult to distinguish situations in which different voices articulate registrally equivalent arrangements of notes. In this paper, we will assume that this cannot be done, considering only situations in which voices are ordered in register.

Fig. 3. The three pairwise swaps that generate the subgroup of voice exchanges; these swap root and fifth, third and fifth, and root and third.

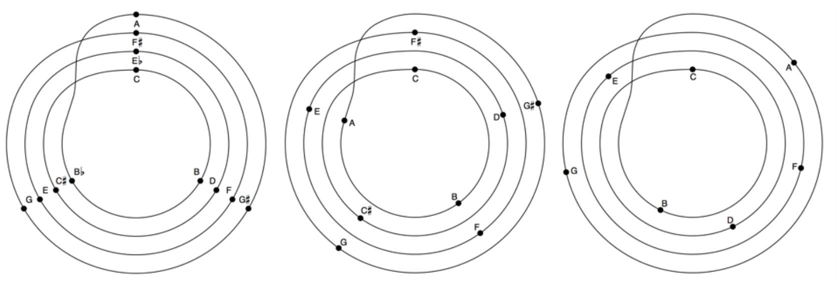

Fig. 4. The crossing-free voice leadings for four-note chords in twelve-note, ten-note, and seven-note space.

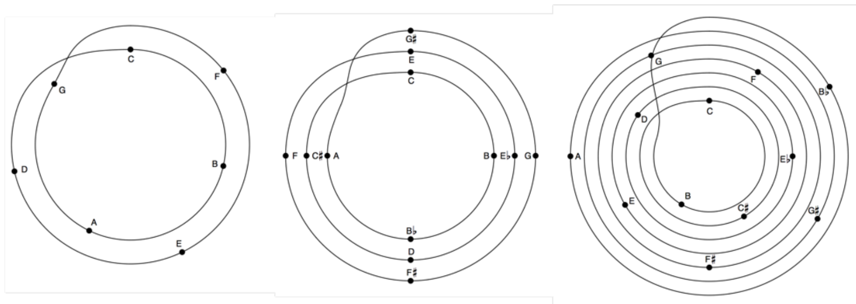

Fig. 5. The crossing-free voice leadings for two-note chords in seven-note space, three-note chords in twelve-note space, and seven-note chords in twelve-note space.

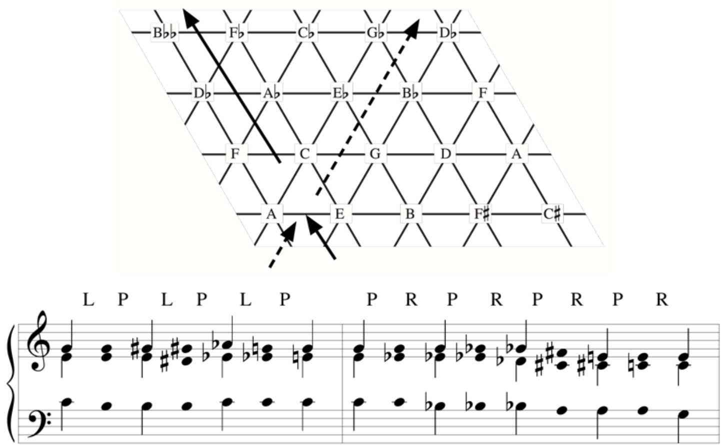

Fig. 6. The LP cycle (solid line) and PR cycle (dotted line) look similar on the Tonnetz, but are musically very different, the former returning all voices to their starting point and the latter acting as a one-step scalar transposition.

Fig. 7. A collection of zero-sum voice leadings that move the doubled note from root to third to fifth and back.

Fig. 8. An open-position version of the music in Fig. 1.

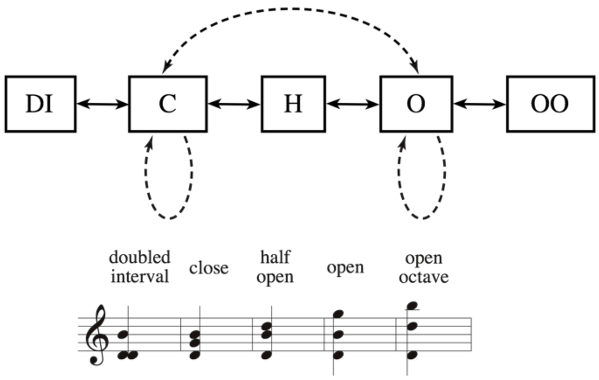

Fig. 9. Five upper-voice configurations.

Fig. 10. DI can be transformed into H, and H into OO, by moving the middle voice up an octave. The same is true of C and O.

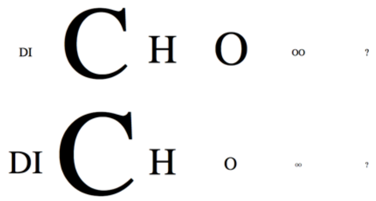

Fig. 11. Distribution of upper-voice configurations in Bach (top) and Palestrina (bottom). Character size is proportional to frequency of occurrence.

Voice leading	**% of total**
C→C	31.8
O→O	11.8
C→H	9.1
H→C	8.5
H→O	4.6
O→H	4.3
C→DI	3.6
DI→C	3.5
H→H	3.3
O→C	3.1
C→O	2.8
OO→O	1.7
O→OO	1.6

Fig. 12. The most common transitions in the chorales.

Fig. 13. We can transpose the source and destination chord to C without changing the upper-voice configurations.

Fig. 14. We can move the doubling to C without changing the upper-voice configuration.

crossings	**total**	**performable**	**unperformable**	**pct.**
0	4	4	0	100%
1	16	16	0	100%
2	40	40	0	100%
3	80	80	0	100%
4	136	112	24	82%
5	208	128	80	62%
6	296	128	168	43%
7	400	128	272	32%
8	520	128	392	25%
9	656	128	528	20%
10	808	128	680	16%

Fig. 15. The number of performable and unperformable voice leadings, with various numbers of crossings for a four-note chord with no doublings.

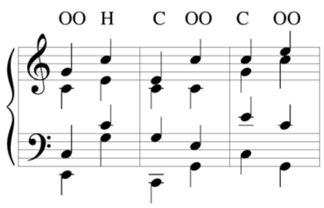

Fig. 16. Three two-crossing voice leadings that are not performable using just C, H, and O.

crossings	**count**	**C**	**C, O**	**C, H, O**	**all**
0	4	75%	75%	100%	100%
1	9	22.2	100	100	100
2	20	10	45	85	100
3	25	0	16	48	100
4	44	4.5	13.6	36.4	86.4
5	59	3.4	10.2	27.1	74.6
6	85	2.4	9.4	23.5	58.8
7	109	1.8	7.3	16.5	47.7
8	140	1.4	5.7	12.9	35.7
9	175	1.1	4.6	11.4	28.6
10	216	0.9	3.7	8.3	24.1

Fig. 17. The proportion of voice leadings with various numbers of crossings performable with various upper-voice configurations.